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RESEARCH ARTICLE

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A semi-analytic method for valuing high-dimensional options on the maximum and minimum of multiple assets

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Abstract Valuing high-dimensional options has many important applications in finance but when the true distributions are unknown or complex, numerical approximations must be used. Approximation methods based on Monte-Carlo simulation show a steep trade-off between estimation accuracy and computational efficiency. This article presents an alternative semi-analytic approximation method for pricing options on the maximum or minimum of multiple assets with unknown distributions. Computational efficiency is shown to improve significantly without sacrificing estimation accuracy. The method is illustrated with applications to options on underlying assets with mean-reverting prices, time-dependent correlations, and stochastic volatility.

Keywords High-dimensional options · Maximum · Minimum · Mean-reverting · Stochastic volatility

JEL Classification Numbers C63 · G12

1 Introduction

As discussed by Stulz (1982), options on the maximum or the minimum of multiple assets have many applications in financial management, such as "the valuation of

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foreign currency debt, option-bonds, compensation plans, risk-sharing contracts, secured debt, and growth opportunities involving mutually exclusive investments" (Stulz 1982, p. 161). Therefore, they are important to investment, financing, and governance decisions of the firm.

The literature on valuing high-dimensional options on the maximum and minimum of multiple underlying assets starts with Margrabe (1978) and Stulz (1982) who developed analytical solutions to the valuation of European maximum and minimum options on two assets with lognormal processes. Stulz's (1982) solution was generalized by Johnson (1987) to an arbitrary number of underlying assets assuming a cumulative multivariate normal distribution. Asset return processes may not be lognormal or even unknown in many applications. For unknown processes, Boyle and Tse (1990) proposed an approximation approach by applying Clark's (1961) algorithm for calculating the moments of the stochastic process for the maximum or minimum of a finite set of random variables. By not specifying the distribution, this approximation approach is also applicable to assets that do not have lognormal processes.

The trade-off between estimation accuracy and computational efficiency is well known for Monte-Carlo simulation. When the underlying assets have high volatilities, stochastic volatilities, or time-dependent correlations or when the option has a long maturity or high dimensionality, precision deterioration becomes a major concern unless computation time is increased exponentially.

This article presents a simple semi-analytic, approximation approach that provides for an accurate and computationally efficient valuation of European options on the maximum or minimum of an arbitrary number of underlying assets. The essence of this approach is to approximate the stochastic process generated by the maximum or the minimum of multiple asset processes with an artificial single dimensional lognormal process by applying Clark's (1961) algorithm. The well-known Black–Scholes formula is then used to value the option on this single dimension lognormal process. The gain in computational efficiency arises from eliminating the need for simulation; this is achieved by converting the multidimensional problem to one with a single dimension and approximating this single dimensional process with a lognormal process. Computation time is typically reduced to less than 0.02 s. Compared with the approach used by Boyle and Tse (1990), there is negligible difference in accuracy.

To further illustrate the applicability of the approach, it is applied to assets with mean-reverting prices, time-varying correlations, and stochastic volatilities. Mean-reverting behavior is generally assumed for interest rates and energy prices.[1] Robustness tests were conducted for options with long maturities and higher dimensionality.

This study makes several contributions. First, the proposed approach improves computational efficiency dramatically with little deterioration in estimation pre-

[1] As stated by Deng (2005), Ronald and Mahieu (2001), and Geman and Roncoroni (2006), for instance, the electricity spot prices are very complicated due to the existence of time-varying correlations and stochastic volatility. Furthermore, the volatility of electricity markets can be as high as 100%, and the expected return can even be negative because of the nonstorability and nontradability. Therefore, the electricity spot prices should be determined by other derivative assets in the market, but not by the underlying assets directly due to the nontradability and nonstorability of electricity. More adjustments, such as using forward prices (Koekebakker and Ollmar 2001; Audet, Heiskanen, Keppo, and Vehvilainen, 2004), need to be made to the general approach to fit the special nature of the electricity market. The authors thank anonymous referees for pointing out this issue.

cision when valuing high-dimensional maximum or minimum options. Second, numerical examples suggest that the approach is robust and stable even when the options have high dimensionality or long maturity and the assets have high and/or stochastic volatility, or time-dependent correlation among them. These valuation applications have not been demonstrated before. Third, this adds to the literature on option valuation when the underlying asset's stochastic process is not lognormal or unknown. Fourth, the approach is easy to implement, giving it the potential of wide adoption by academics and practitioners.

The article is organized as follows: Section 2 presents the two-step semi-analytic approach and illustrates it with examples from Boyle and Tse (1990). Section 3 applies the approach to options on underlying assets with mean-reverting prices while Section 4 applies it to options on assets with stochastic volatilities. Concluding remarks are made in Section 5.

2 The two-step valuation approach

This approach for valuing a high-dimensional maximum or minimum European option on multiple assets with unknown distributions is based, as in Boyle and Tse (1990), on Clark's (1961) method for calculating the moments of the maximum or minimum of a finite set of random variables. It reduces the multidimensional problem to one with a single dimension and then approximates the single dimensional process with the lognormal process. As a result, the Black–Scholes formula can be applied directly, simulation is avoided, and computational efficiency is achieved.

2.1 Step 1: approximate moments via an artificial lognormal process

We consider m $(m > 1)$ underlying assets, $x_1, x_2, \ldots, x_m$, with unknown distributions. Each underlying asset x_i $(i = 1, \ldots, m)$ follows a generalized form

$$\begin{cases} dx_i(t) = \alpha_i(t)x_i(t)dt + b_i(t,x)dz_i(t), \\ x_i(0) = x_{i0}, \end{cases} \tag{1}$$

where $\alpha_i(t)$ is the drift and $b_i(t,x)$ is the volatility.

$$z(t) := (z_1(t), z_2(t), \ldots, z_m(t))$$

is an m-dimensional Brownian motion with a given correlation matrix

$$\rho(t) := (\rho_{ij}(t)).$$

Thus, the payoff of the *maximum call option* and that of the *maximum put option* are

$$\mathrm{Payoff}_{C_{\max}}(\mathbf{x}) = \hat{E}[\max(\max(x_1, x_2, \ldots, x_m) - K, 0)], \tag{2}$$

$$\mathrm{Payoff}_{P_{\max}}(\mathbf{x}) = \hat{E}[\max(K - \max(x_1, x_2, \ldots, x_m), 0)], \tag{3}$$

respectively, where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is an m -dimensional vector, K is the strike price, and $\hat{E}$ is the expectation. Similarly, the payoff of the *minimum call* *option* and that of the *minimum put option* with the same strike price K are, respectively,

$$\text{Payoff}_{C_{\min}}(\mathbf{x}) = \hat{E}[\max(\min(x_1, x_2, \ldots, x_m) - K, 0)], \tag{4}$$

$$\text{Payoff}_{P_{\min}}(\mathbf{x}) = \hat{E}[\max(K - \min(x_1, x_2, \ldots, x_m), 0)]. \tag{5}$$

The process $x_i(\cdot)$ is expressed via a lognormal process $S_i(\cdot)$ in every small interval $[t_{\ell-1}, t_\ell]$ for $\ell = 1, \cdots, n$

$$\begin{cases} dS_i(t) = (r(t) - q_i(t))S_i(t)dt + \sigma_i(t)S_i(t)dz_i(t), \\ S_i(0) = x_{i0}, \end{cases} \tag{6}$$

where

$$\begin{cases} q_i(t) = q_{i\ell}, \ \forall t \in [t_{\ell-1}, t_\ell], \\ \sigma_i(t) = \sigma_{i\ell}, \ \forall t \in [t_{\ell-1}, t_\ell], \quad \text{for } \ell = 1, 2, \ldots, n. \end{cases}$$

To do this, we divide $[0, T]$ into finitely many small periods $[t_{\ell-1}, t_\ell]$, where $\Delta t_\ell = t_\ell - t_{\ell-1}$ for $\ell = 1, \ldots, n$. In every one of these small intervals $[t_{\ell-1}, t_\ell]$, (6) gives

$$\begin{cases} \hat{E}S_i(t_\ell) = \hat{E}S_i(t_{\ell-1})e^{\int_{t_{\ell-1}}^{t_\ell} r(t)dt - q_{i\ell}\Delta t_\ell}, \\ \hat{E}S_i(t_\ell)^2 = \hat{E}S_i(t_{\ell-1})^2 e^{\int_{t_{\ell-1}}^{t_\ell} 2r(t)dt + (\sigma_{i\ell}^2 - 2q_{i\ell})\Delta t_\ell}. \end{cases}$$

Re-arranging the terms of the above equation yields

$$\begin{cases} q_{i\ell} = \frac{1}{\Delta t_\ell}\left[\int_{t_{\ell-1}}^{t_\ell} r(t)dt - \ln\left(\frac{\hat{E}S_i(t_\ell)}{\hat{E}S_i(t_{\ell-1})}\right)\right], \\ \sigma_{i\ell} = \sqrt{\frac{1}{\Delta t_\ell}\left[\ln\left(\frac{\hat{E}S_i(t_\ell)^2}{\hat{E}S_i(t_{\ell-1})^2}\right) - 2\ln\left(\frac{\hat{E}S_i(t_\ell)}{\hat{E}S_i(t_{\ell-1})}\right)\right]}. \end{cases}$$

Setting the first and second moments of the log-normal process $S_i(\cdot)$ equal to those of the process $x_i(\cdot)$ at the time points $t_{\ell-1}$ and t_ℓ in every sufficiently small interval $[t_{\ell-1}, t_\ell]$, we obtain

$$\begin{cases} q_{i\ell} = \frac{1}{\Delta t_\ell}\left[\int_{t_{\ell-1}}^{t_\ell} r(t)dt - \ln\left(\frac{\hat{E}x_i(t_\ell)}{\hat{E}x_i(t_{\ell-1})}\right)\right], \\ \sigma_{i\ell} = \sqrt{\frac{1}{\Delta t_\ell}\left[\ln\left(\frac{\hat{E}x_i(t_\ell)^2}{\hat{E}x_i(t_{\ell-1})^2}\right) - 2\ln\left(\frac{\hat{E}x_i(t_\ell)}{\hat{E}x_i(t_{\ell-1})}\right)\right]}, \end{cases}$$

where $\hat{E}x_i(t)$ and $\hat{E}x_i(t)^2$ are the first and the second moments, respectively, as defined above.

Let $X_i = \ln S_i$, $i = 1, 2, \ldots, m$, be the ith amongst m jointly normal variates according to asset prices defined in (6). Applying Itô's lemma to $X_i = \ln S_i$ yields

$$\begin{cases} dX_i(t) = (r(t) - q_i(t) - \frac{1}{2}\sigma_i(t)^2)dt + \sigma_i(t)dz_i(t), \\ X_i(0) = \ln S_i(0). \end{cases}$$

We also have the expected value $\zeta_i(t)$ and variance $\eta_i(t)^2$ of $X_i(\cdot)$ at time t via a simple calculation

$$\begin{cases} \zeta_i(t) = \int_0^t (r(s) - q_i(s) - \frac{1}{2}\sigma_i(s)^2)ds + \ln S_i(0), \\ \eta_i(t)^2 = \int_0^t \sigma_i(s)^2 ds. \end{cases} \tag{7}$$

Due to the monotonicity of the exponential function, in addition, we obtain

$$\begin{aligned} \max(S_1, S_2, \ldots, S_m) &= \max(\exp(X_1), \exp(X_2), \ldots, \exp(X_m)) \\ &= \exp(\max(X_1, X_2, \ldots, X_m)) \\ &= \exp(Y), \end{aligned}$$

where $Y = \max(X_1, X_2, \ldots, X_m)$. Extending Clark's algorithm to a time-dependent process, we can derive the expected value $\zeta(t)$ and variance $\eta(t)^2$ of the process $Y(\cdot)$[2] according to (7).

Assuming that $Y(\cdot)$ is normally distributed with mean $\zeta(t)$ and variance $\eta(t)^2$, this provides us with a way to convert the high-dimensional process to a lognormal process by setting $S = \exp(Y)$, which satisfies

$$\begin{cases} dS(t) = S(t)\{(r(t) - \delta(t))dt + \nu(t)dz(t)\}, \\ S(0) = \max(S_1(0), S_2(0), \ldots, S_m(0)), \end{cases}$$

where $\delta(t)$ and $\nu(t)$ are the dividend and volatility of the artificial security $S(\cdot)$ at time t, respectively, in the transformed setting. Its initial price is determined based on the instantaneous price.[3]

[2] At $t = 0$,

$$Y(0) = \max(X_1(0), X_2(0), \ldots, X_m(0)) = \max(\ln S_1(0), \ln S_2(0), \ldots, \ln S_m(0)).$$

Due to the monotonicity of the logarithm function, we obtain

$$Y(0) = \ln(\max(S_1(0), S_2(0), \ldots, S_m(0)))$$

and

$$Y(0)^2 = [\ln(\max(S_1(0), S_2(0), \ldots, S_m(0)))]^2.$$

Then

$$\zeta(0) = \ln(\max(S_1(0), S_2(0), \ldots, S_m(0)))$$

and

$$\eta(0)^2 = 0$$

[3] Suppose this artificial security to be sold immediately within Δt after it is purchased. Then

$$S(\Delta t) = \max(S_1(\Delta t), S_2(\Delta t), \ldots, S_m(\Delta t)),$$

and therefore,

$$\lim_{\Delta t \to 0} S(\Delta t) = \max(S_1(0), S_2(0), \ldots, S_m(0)).$$

The value $\lim_{\Delta t \to 0} S(\Delta t)$ is considered the initial price of the fictitious security process $S(\cdot)$

Applying the Itô's formula again to $Y = \ln S$ yields

$$\begin{cases} dY(t) = (r(t) - \delta(t) - \frac{1}{2}\nu(t)^2)dt + \nu(t)dz(t), \\ Y(0) = \ln S(0). \end{cases}$$

Therefore,

$$\begin{cases} \int_0^t (r(s) - \delta(s) - \frac{1}{2}\nu(s)^2)ds + \ln S(0) = \zeta(t), \\ \int_0^t \nu(s)^2 dt = \eta(t)^2. \end{cases} \tag{8}$$

Since the only two variables that need to be determined are $\delta(t)$ and $\nu(t)$ in the equation system (8) in every small interval $[t_{\ell-1}, t_\ell]$, we can write

$$\begin{cases} \delta(t) = \delta_\ell, \\ \nu(t) = \nu_\ell, \ \forall t \in [t_{\ell-1}, t_\ell], \quad \text{for } \ell = 1, 2, \ldots, n, \end{cases}$$

This means, according to (8),

$$\begin{cases} \delta_\ell = \frac{1}{\Delta t_\ell}(\zeta(t_{\ell-1}) - \zeta(t_\ell) + \int\limits_{t_{\ell-1}}^{t_\ell} r(s)ds) - \frac{1}{2}\nu_\ell^2, \\ \nu_\ell^2 = \frac{1}{\Delta t_\ell}(\eta(t_\ell)^2 - \eta(t_{\ell-1})^2), \end{cases}$$

where $\Delta t_\ell = t_\ell - t_{\ell-1}$.

Similarly, the minimum of multiple risky assets, $\min(S_1, S_2, \ldots, S_m)$, can be transformed to

$$\begin{aligned} \min(S_1, \ldots, S_m) &= \min(\exp(X_1), \exp(X_2), \ldots, \exp(X_m)) \\ &= \exp(\min(X_1, X_2, \ldots, X_m)) \\ &= \exp(-\max(-X_1, -X_2, \ldots, -X_m)) \\ &= \exp(-\overline{Y}), \end{aligned}$$

where $\overline{Y} = \max(-X_1, -X_2, \ldots, -X_m)$,[4] as a preparation for the valuation algorithm. Extending to a time-dependent Clark algorithm as above, we can derive the expected value $\overline{\zeta}(t)$ and variance $\overline{\eta}(t)^2$ of the process $\overline{Y}(\cdot)$. Based on that, a simple calculation will provide the expected value $-\overline{\zeta}(t)$ and the variance $\overline{\eta}(t)^2$ of the process $-\overline{Y}(\cdot)$.

Thus, the artificial lognormal security process $\overline{S} = \exp(-\overline{Y})$ for the minimum of risky assets satisfies

$$\begin{cases} d\overline{S}(t) = \overline{S}(t)\{(r(t) - \overline{\delta}(t))dt + \overline{\nu}(t)dz(t)\}, \\ \overline{S}(0) = \min(S_1(0), S_2(0), \ldots, S_m(0)), \end{cases}$$

with dividend $\overline{\delta}(t)$ and volatility $\overline{\nu}(t)$ at time t in the transformed setting. Similar to the one for the maximum of risky assets, the initial price of the artificial security for the minimum of risky assets, $\overline{S}(0) = \min(S_1(0), S_2(0), \ldots, S_m(0))$, is determined by the instantaneous price.

[4] This is similar to Boyle and Tse's (1990) functional form for the same situation.

In every sufficiently small interval $[t_{\ell-1}, t_\ell]$, we obtain

$$\begin{cases} \overline{\delta}(t) = \overline{\delta}_\ell, \\ \overline{\nu}(t) = \overline{\nu}_\ell, \ \forall t \in [t_{\ell-1}, t_\ell], \quad \text{for } \ell = 1, 2, \ldots, n, \end{cases}$$

with

$$\begin{cases} \overline{\delta}_\ell = \frac{1}{\Delta t_\ell}(\overline{\zeta}(t_{\ell-1}) - \overline{\zeta}(t_\ell) + \int\limits_{t_{\ell-1}}^{t_\ell} r(s)ds) - \frac{1}{2}\overline{\nu}_\ell^2, \\ \overline{\nu}_\ell^2 = \frac{1}{\Delta t_\ell}(\overline{\eta}(t_\ell)^2 - \overline{\eta}(t_{\ell-1})^2), \end{cases}$$

where $\Delta t_\ell = t_\ell - t_{\ell-1}$, by applying the same techniques presented above.

2.2 Step 2: a semi-analytic expression for pricing high-dimensional maximum and minimum options

Once the multi-dimensional problem has been reduced and approximated by a single dimensional lognormal process, we are ready to value the high-dimensional maximum or minimum option. For the artificial security $S(\cdot)$ constructed in section 2.1, the value of a European maximum call option on that security, according to the payoff described in (2), is

$$C_{\max}(\mathbf{x}) = e^{-\int_0^T r(t)dt} \hat{E}[\max(S - K, 0)].$$

Applying the one-dimensional Black–Scholes pricing formula yields

$$C_{\max}(\mathbf{x}) = S(0)e^{-\int_0^T \delta(t)dt}\Phi(d_1) - Ke^{-\int_0^T r(t)dt}\Phi(d_2),$$

where

$$\begin{cases} d_1 = \dfrac{\ln S(0) - \ln K + \int_0^T (r(t) - \delta(t) + \frac{1}{2}\eta(t)^2)dt}{\sqrt{\int_0^T \eta(t)^2 dt}}, \\ d_2 = d_1 - \sqrt{\int_0^T \eta(t)^2 dt}. \end{cases} \tag{9}$$

Substituting (8) into (9) yields

$$\begin{cases} d_1 = \dfrac{\zeta(T) - \eta(T)^2 - \ln K}{\eta(T)}, \\ d_2 = \dfrac{\zeta(T) + \eta(T)^2 - \ln K}{\eta(T)}, \end{cases} \tag{10}$$

and

$$S(0)e^{-\int_0^T \delta(t)dt} = e^{-\int_0^T r(t)dt + \zeta(T) + \frac{1}{2}\eta(T)^2}. \tag{11}$$

One realizes that neither (10) nor (11) requires the third or the fourth moment, and therefore, it supports our argument about their negligible influence. In section

3, we will show that, using numerical examples, this does not hurt the accuracy of our valuation approach.

Applying the same procedure, the value of a European minimum put option is

$$P_{\min}(\mathbf{x}) = e^{-\int_0^T r(t)dt}\hat{E}[\max(K-\overline{S},0)],$$

based on (5) and the artificial security process $\overline{S}(\cdot)$ for the minimum of multiple risky assets. Applying the Black–Scholes formula again, therefore, the valuation algorithm for the minimum put option is

$$P_{\min}(\mathbf{x}) = Ke^{-\int_0^T r(t)dt}\Phi(-\overline{d}_2) - \overline{S}(0)e^{-\int_0^T \overline{\delta}(t)dt}\Phi(-\overline{d}_1),$$

where

$$\begin{cases} \overline{d}_1 = \dfrac{\overline{\zeta}(T) - \overline{\eta}(T)^2 - \ln K}{\overline{\eta}(T)}, \\ \overline{d}_2 = \dfrac{\overline{\zeta}(T) + \overline{\eta}(T)^2 - \ln K}{\overline{\eta}(T)}, \\ \overline{S}(0)e^{-\int_0^T \overline{\delta}(t)dt} = e^{-\int_0^T r(t)dt + \overline{\zeta}(T) + \frac{1}{2}\overline{\eta}(T)^2}. \end{cases}$$

It is interesting to note that both formulas involve only the first and second moments. This is one of the justifications for ignoring the higher moments in this approach.

2.3 A summary of the proposed two-step semi-analytic valuation approach

The two-step semi-analytic valuation approach presented above may be summarized as follows:

- Divide the period $[0, T]$ into finitely many small intervals $[t_{\ell-1}, t_\ell]$;
- Introduce an artificial lognormal process $S_i(\cdot)$, whose initial price $S_i(0)$ is set equal to x_{i0}, $i = 1, 2, \ldots, m$, and calculate the first and second moments of the lognormal process $S_i(\cdot)$ within each of the small intervals $[t_{\ell-1}, t_\ell]$ using equation (6);
- Equate the first and second moments of $S_i(\cdot)$ and those of $x_i(\cdot)$, respectively, in each sufficiently small interval $[t_{\ell-1}, t_\ell]$;
- Reduce the dimensionality by extending Clark's original algorithm to a time-dependent situation and converting the high-dimensional process to one with a single dimension using equation (7);
- Apply the Black–Scholes formula to $S(\cdot)$.

This approach is different from that of Boyle and Tse (1990) in two aspects. First, the arbitrary dimensionality is reduced to one by the approximation algorithm. Second, the single dimensional process is approximated by a lognormal process making it possible to apply the Black–Scholes formula and eliminating the need for simulation. In the next section, we illustrate the approach by applying it to the examples in Boyle and Tse (1990). Then we apply it to a selection of other processes in the remaining sections.

2.4 Illustrations of the accuracy of the approach

To illustrate the estimation accuracy of the proposed approximation method for valuing European options on the maximum or the minimum of multiple risky assets, we first replicate the examples adopted by Boyle and Tse (1990) and compare the accuracy of our results with theirs. We follow Boyle and Tse (1990) by treating the simulation results in Boyle (1989) and Johnson (1987) as benchmarks for determining accuracy. The results are presented in Tables 1–4. Boyle and Tse (1990) do not present data on the computation times for their approach. Therefore, it is not possible to compare our computational efficiency with theirs.

Table 1 shows the results by applying the approach to Boyle and Tse's (1990) hypothetical example which assumes that all risky underlying assets have the same initial price of \$40, the same volatility of 25% per year, and the same correlation coefficient of 0.95 with other assets. When the strike price is assumed to be zero and the continuously-compounded risk-free rate is fixed at 10% per annum, the values of nine-month options on the minimum of up to 50 assets obtained using our approximation method are almost identical to those using Boyle and Tse's (1990) method. In other words, treating simulation results in the existing literature (Boyle 1989) as benchmarks shows that the estimation accuracy of the method proposed in the current study is as good as that of Boyle and Tse; it also confirms

Table 1 Comparison of European call options on the minimum of multiple assets with benchmarks provided by Boyle's (1989) valuation and with results from Boyle and Tse's (1990) valuation

Number of assets	Correlation coefficient = 0.95				Correlation coefficient = 0.995		
	Options			Relative error(%)	Options		Relative error(%)
	LW	BT	Boyle		LW	Boyle	
2	38.908	38.908	38.908	0.000	39.655	39.654	0.003
3	38.371	38.371	38.374	−0.008	39.482	39.483	−0.003
4	38.029	38.029	38.033	−0.011	39.371	39.372	−0.003
5	37.782	37.782	37.786	−0.011	39.291	39.292	−0.003
10	37.102	37.102	37.100	0.005	39.067	39.066	0.003
15	36.753	36.752	36.746	0.016	38.951	–	–
20	36.522	36.522	36.511	0.030	38.873	38.869	0.010
25	36.351	36.351	36.338	0.036	38.816	–	–
30	36.217	36.217	36.201	0.044	38.771	38.765	0.016
35	36.107	36.107	36.089	0.050	38.734	–	–
40	36.014	36.013	35.994	0.056	38.702	38.695	0.018
45	35.933	35.933	35.912	0.059	38.675	–	–
50	35.862	35.862	35.840	0.061	38.650	38.643	0.018

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by BT and Boyle present option prices from Boyle and Tse (1990) and Boyle (1989), respectively. The relative error is measured by the difference between results obtained from the proposed method and those in Boyle (1989).
For each underlying asset:
Time to maturity: $T = 0.75$ year
Risk-free rate: 10% per annum, continuously compounded
Volatilities: $v_i = 25\%$ per annum
Spot prices at t_0: $x_i(0) = 40$

Table 2 Comparison of European call options on the maximum and the minimum of three assets with benchmarks and with results from Boyle and Tse's (1990) valuation

	Assets		
	One	Two	Three
Current prices	40	40	40
Volatilities(%)	30	30	30
Correlation	1.0	0.9	0.9
Matrix	0.9	1.0	0.9
	0.9	0.9	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	16.353	16.351	16.351	0.012	10.398	10.396	10.405	−0.067
35	12.385	12.383	12.384	0.008	7.087	7.086	7.094	−0.099
40	8.986	8.984	8.986	0.000	4.583	4.581	4.588	−0.109
45	6.268	6.267	6.270	−0.032	2.836	2.835	2.840	−0.141
50	4.226	4.226	4.229	−0.071	1.695	1.694	1.698	−0.177

	Assets		
	One	Two	Three
Current prices	40	40	40
Volatilities(%)	30	30	30
Correlation	1.0	0.6	0.4
Matrix	0.6	1.0	0.6
	0.4	0.6	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	20.046	20.046	20.018	0.140	7.184	7.184	7.214	−0.416
35	15.762	15.758	15.730	0.203	4.330	4.323	4.345	−0.345
40	11.861	11.855	11.832	0.245	2.422	2.408	2.419	0.124
45	8.538	8.536	8.520	0.211	1.275	1.259	1.262	1.030
50	5.895	5.901	5.895	0.000	0.642	0.626	0.626	2.556

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by BT and Johnson present option prices from Boyle and Tse (1990) and Johnson (1987), respectively. The relative error is measured by the difference between results obtained from the proposed method and those in Johnson (1987)

that the estimation error increases with the number of underlying assets (Boyle and Tse 1990). Results with higher correlation coefficients also show that the valuation error decreases as the correlation coefficient rises.

We also replicated the other set of three-asset examples presented in Boyle and Tse (1990). They include six combinations of variates in three factors: initial price, volatility, and pairwise correlation coefficients. Following Boyle and Tse (1990), accuracy was based on the prices of European call options on the maximum or the minimum obtained by Johnson (1987). Table 2 presents a comparison of values obtained by the two approaches for call options when assets have the same initial price and volatility. The results in Table 3 assume unequal initial prices and those in Table 4 assume unequal volatilities.

Table 3 Comparison of European call options on the maximum and the minimum of three assets with benchmarks and with results from Boyle and Tse's (1990) valuation

	Assets: One	Two	Three
Current prices	40	45	50
Volatilities(%)	30	30	30
Correlation	1.0	0.9	0.9
Matrix	0.9	1.0	0.9
	0.9	0.9	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	23.749	23.748	23.765	−0.067	12.679	12.677	12.686	−0.055
35	19.433	19.431	19.448	−0.077	9.078	9.076	9.084	−0.066
40	15.422	15.419	15.436	−0.091	6.196	6.194	6.200	−0.065
45	11.868	11.866	11.882	−0.118	4.058	4.057	4.061	−0.074
50	8.873	8.872	8.888	−0.169	2.570	2.569	2.570	0.000

	Assets: One	Two	Three
Current prices	40	45	50
Volatilities(%)	30	30	30
Correlation	1.0	0.6	0.4
Matrix	0.6	1.0	0.6
	0.4	0.6	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	26.950	26.954	26.955	−0.019	9.900	9.903	9.973	−0.732
35	22.511	22.511	22.510	0.004	6.540	6.538	6.600	−0.909
40	18.253	18.248	18.245	0.044	4.043	4.031	4.078	−0.858
45	14.333	14.325	14.321	0.084	2.363	2.342	2.373	−0.421
50	10.896	10.891	10.889	0.064	1.321	1.296	1.314	0.533

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by BT and Johnson present option prices from Boyle and Tse (1990) and Johnson (1987), respectively. The relative error is measured by the difference between results obtained from the proposed method and those in Johnson (1987)

From Table 2, one sees that, when all three assets have equal initial prices, equal volatilities and symmetric pairwise correlations, our valuation method gives better approximations than Boyle and Tse's (1990) in 10 out of 20 examples, indicating no advantage in accuracy for either approach. Relative errors generated by the two approaches are, however, both acceptable except for low priced options ($< 2.6\%$).

When the assumption on the equality of asset initial prices is relaxed, the results, as shown in Table 3 are again similar. Our valuations are slightly better than those obtained by Boyle and Tse's (1990) (14 out of 20 examples). When the assumption on the equality of volatilities is relaxed, Table 4 shows that Boyle and Tse's method is more accurate in 17 out of the 20 examples. In general, both approximation methods yield more accurate results with in-the-money options than with

Table 4 Comparison of European call options on the maximum and the minimum of three assets with benchmarks and with results from Boyle and Tse's (1990) valuation

	Assets: One	Two	Three
Current prices	40	40	40
Volatilities(%)	25	30	35
Correlation	1.0	0.9	0.9
Matrix	0.9	1.0	0.9
	0.9	0.9	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	16.705	16.703	16.687	0.108	10.167	10.172	10.178	−0.108
35	12.701	12.682	12.661	0.316	6.944	6.914	6.917	0.390
40	9.249	9.235	9.223	0.282	4.516	4.441	4.427	2.010
45	6.473	6.490	6.496	−0.354	2.823	2.715	2.681	5.259
50	4.375	4.438	4.462	−1.950	1.710	1.593	1.545	10.680

	Assets: One	Two	Three
Current prices	40	40	40
Volatilities(%)	25	30	35
Correlation	1.0	0.6	0.4
Matrix	0.6	1.0	0.6
	0.4	0.6	1.0

Strike price	Call on maximum: Options LW	BT	Johnson	Relative error(%)	Call on minimum: Options LW	BT	Johnson	Relative error(%)
30	20.178	20.185	20.153	0.124	7.177	7.172	7.172	0.070
35	15.893	15.885	15.847	0.290	4.369	4.326	4.311	1.345
40	11.990	11.969	11.929	0.511	2.483	2.409	2.379	4.372
45	8.660	8.644	8.611	0.569	1.336	1.258	1.220	9.508
50	6.004	6.013	5.995	0.150	0.689	0.625	0.589	16.978

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by BT and Johnson present option prices from Boyle and Tse (1990) and Johnson (1987), respectively. The relative error is measured by the difference between results obtained from the proposed method and those in Johnson (1987)

out-of-the-money ones. When the option value falls below \$5, the relative error is above 1% and is unacceptable at 16.98% when the option value is below \$1.

3 Applications to assets with mean-reverting prices

Many applications in financial management involve assets with mean-reverting processes. For example, interest rates are now commonly modelled as mean-reverting processes. In this section, we apply the approach to options on the maximum or minimum of m $(m > 1)$ underlying assets with mean-reverting prices. The process $x_i(\cdot)$ follows the general H–W model (Hull and White 1994a,b)

$$\begin{cases} dx_i(t) = \kappa_i(t)(\gamma_i(t) - x_i(t))dt + \nu_i(t)x_i(t)^\beta dz_i(t), \\ x_i(0) = x_{i0}, \end{cases} \tag{12}$$

where $\gamma_i(t)$ and $\nu_i(t)$ are the long-run mean and the volatility of the ith mean-reverting asset x_i, respectively, and β is a constant. $\kappa_i(t)$ is the strength of mean-reversion, which determines the speed of adjustment pulling the randomly moving value of the ith mean-reverting asset x_i toward a central location $\gamma_i(t)$. $z(t)$ and $\rho(t)$ are defined as in the general economic setting described above. We then value a European call or put option on the maximum or the minimum of multiple assets as described in section 2.

For analytical simplicity, we assume that parameters $\kappa_i(t)$, $\gamma_i(t)$, and $\nu_i(t)$, as well as the short rate $r(t)$, are deterministic following Duffie and Richardson (1991). A direct calculation yields the first moment of the general H–W model (12)

$$\hat{E}x_i(t) = e^{-\int_0^t \kappa_i(s)ds}\Big(x_{i0} + \int_0^t \kappa_i(s)\gamma_i(s)e^{\int_0^s \kappa_i(u)du}ds\Big),$$

where $\hat{E}$ indicates an expectation.

In the finance literature, the Vasicek–Ornstein–Uhlenbeck (VOU) model (Vasicek 1977) where $\beta = 0$ in (12), the Cox–Ingersoll–Ross (CIR) model (Cox, Ingersoll and Ross 1981; 1985) where $\beta = \frac{1}{2}$, and the exponential Ornstein–Uhlenbeck (EOU) model where $\beta = 1$, are the three most commonly used mean-reverting models. They have different expressions of their second moments,

$$\begin{aligned}\hat{E}x_i(t)^2 &= e^{-\int_0^t 2\kappa_i(s)ds}\int_0^t \nu_i(s)^2 e^{\int_0^s 2\kappa_i(u)du}ds \\ &\quad + e^{-\int_0^t 2\kappa_i(s)ds}\Big(x_{i0} + \int_0^t \kappa_i(s)\gamma_i(s)e^{\int_0^s \kappa_i(u)du}ds\Big)^2, \end{aligned} \tag{13}$$

for the VOU model,

$$\begin{aligned}\hat{E}x_i(t)^2 &= e^{-\int_0^t 2\kappa_i(s)ds}\int_0^t \nu_i(s)^2\Big(x_{i0} + \int_0^s \kappa_i(u)\gamma_i(u)e^{\int_0^u \kappa_i(v)dv}du\Big)e^{\int_0^s \kappa_i(u)du}ds \\ &\quad + e^{-\int_0^t 2\kappa_i(s)ds}\Big(x_{i0} + \int_0^t \kappa_i(s)\gamma_i(s)e^{\int_0^s \kappa_i(u)du}ds\Big)^2, \end{aligned} \tag{14}$$

for the CIR model, and

$$\begin{aligned}\hat{E}x_i(t)^2 &= e^{\int_0^t (\nu_i(s)^2 - 2\kappa_i(s))ds}\int_0^t \nu_i(s)^2\Big(x_{i0} + \int_0^s \kappa_i(u)\gamma_i(u)e^{\int_0^u \kappa_i(v)dv}du\Big)^2 \\ &\quad \times e^{-\int_0^s \nu_i(u)^2du}ds + e^{-\int_0^t 2\kappa_i(s)ds}\Big(x_{i0} + \int_0^t \kappa_i(s)\gamma_i(s)e^{\int_0^s \kappa_i(u)du}ds\Big)^2, \end{aligned} \tag{15}$$

for the EOU model.

We applied the two-step valuation approach to these three models. For the purpose of computational efficiency and estimation accuracy, we compare the results obtained using our method with those using Monte-Carlo simulations as the latter have been widely accepted as accurate estimates of values for European options on assets with mean-reverting prices (Hull 2006).

3.1 Numerical examples

This subsection presents numerical examples to illustrate the computational efficiency and estimation accuracy of the proposed approximation approach in pricing high-dimensional maximum and minimum options on underlying assets with mean-reverting prices. To save space, we only present simulation results for the CIR model; applications based on the VOU and EOU models give similar levels of computational efficiency and accuracy.[5]

In Table 5, we present the results of using the proposed approximation method to value put options on the minimum of and call options on the maximum of two mean-reverting assets. Both assets have the same initial price of \$50, the same volatility of 20% per year, and the same strength of mean-reversion equal to $\frac{1}{3}$. Two different correlation coefficients are used (0.5 and 0.2) but only one risk-free rate (5% per annum, continuously compounded). The time to maturity is either 1 year or 3 years.

The results again show that the approach is, in general, better at valuing in-the-money options. For in-the-money maximum call options with 1-year to maturity, the largest relative error between the simulation result and the Monte-Carlo values is only -0.029% ($K = 45$, $\rho = 0.5$). For in-the-money minimum put options with 1-year to maturity, the largest relative error is 0.033% ($K = 55$, $\rho = 0.5$).

For the out-of-the-money options, the highest relative errors were above 30%. But these relative error numbers may not be as significant as they seem because the option values in these cases were close to zero ($<$ \$0.001).[6]

As discussed previously, the accuracy of the proposed approach is similar to the extant method. Computational efficiency is where the approach makes the most important improvement in option valuation technique. In the cases presented in Table 5, our approach only takes 0.003% of the time required using Monte-Carlo (MC) simulations.[7] The Matlab program for our approximation algorithm needs 0.018 s, while the average round using MC simulation needs about 34.40 s.

3.2 The effects of higher volatility, higher dimensionality, and longer maturity

The computational finance literature shows that accuracy and computational efficiency both deteriorate with higher volatility and longer maturity; this is considered one of the major shortcomings of approximation methods. In valuing high-dimensional options, high dimensionality generates similar computational disadvantages. To test the stability of our two-step semi-analytical method, therefore, we apply it to the higher-dimensional options on the maximum and minimum of multiple

[5] Many more tests were conducted but, to economize on space, these representative cases are presented. Other results and results from VOU and EOU models are available from the authors upon request.

[6] Since they are extremely close to zero, the relative errors in different cases are very unstable and meaningless, and therefore out-of-the-money cases will not be presented in most of the other cases.

[7] In each experiment, 20 runs of the Monte-Carlo algorithm with 100, 000 paths are performed so that the mean and variance obtained are stable. CPU time (in seconds) shown in the tables is for one single run, taken as the average over the 20 runs. All runs are performed on a Pentium-III 0.9 GHz machine with the Matlab environment.

Table 5 European put options on the minimum of and European call options on the maximum of two mean-reverting assets in the CIR model

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Correlation $\rho = 0.5$ and $T = 1$ year											
45	0.000(0.000)	34.40	0.000	0.018	–	30	20.178(0.004)	34.40	20.176	0.018	−0.011
50	0.315(0.002)	34.40	0.312	0.018	−0.96	35	15.423(0.004)	34.40	15.420	0.018	−0.018
55	4.524(0.003)	34.40	4.525	0.018	0.033	40	10.665(0.005)	34.40	10.664	0.018	−0.007
60	9.282(0.004)	34.40	9.281	0.018	−0.006	45	5.910(0.004)	34.40	5.908	0.018	−0.029
65	14.037(0.004)	34.40	14.038	0.018	0.004	50	1.225(0.004)	34.40	1.225	0.018	0.021
70	18.794(0.003)	34.40	18.794	0.018	−0.002	55	0.000(0.000)	34.40	0.000	0.018	−0.071
75	23.549(0.003)	34.40	23.550	0.018	0.003	60	0.000(0.000)	34.40	0.000	0.018	–
Case 2: Correlation $\rho = 0.2$ and $T = 1$ year											
45	0.000(0.000)	34.40	0.000	0.018	–	30	20.300(0.004)	34.40	20.298	0.018	−0.010
50	0.343(0.002)	34.40	0.340	0.018	−0.849	35	15.545(0.004)	34.40	15.542	0.018	−0.017
55	4.646(0.003)	34.40	4.647	0.018	0.020	40	10.787(0.005)	34.40	10.786	0.018	−0.010
60	9.404(0.004)	34.40	9.403	0.018	−0.007	45	6.031(0.003)	34.40	6.030	0.018	−0.028
65	14.159(0.004)	34.40	14.160	0.018	0.002	50	1.319(0.003)	34.40	1.321	0.018	0.149
70	18.916(0.003)	34.40	18.916	0.018	−0.003	55	0.000(0.000)	34.40	0.000	0.018	−30.232
75	23.671(0.003)	34.40	23.672	0.018	0.003	60	0.000(0.000)	34.40	0.000	0.018	–
Case 3: Correlation $\rho = 0.5$ and $T = 3$ years											
45	0.000(0.000)	102.48	0.000	0.018	–	30	22.195(0.004)	102.48	22.190	0.018	−0.025
50	0.001(0.000)	102.48	0.000	0.018	−37.679	35	17.890(0.005)	102.48	17.886	0.018	−0.023
55	0.790(0.003)	102.48	0.793	0.018	0.323	40	13.589(0.004)	102.48	13.583	0.018	−0.048
60	4.776(0.005)	102.48	4.778	0.018	0.050	45	9.283(0.005)	102.48	9.279	0.018	−0.038
65	9.078(0.004)	102.48	9.081	0.018	0.036	50	4.980(0.005)	102.48	4.976	0.018	−0.092
70	13.380(0.005)	102.48	13.385	0.018	0.037	55	0.934(0.004)	102.48	0.931	0.018	−0.346
75	17.684(0.005)	102.48	17.688	0.018	0.023	60	0.002(0.000)	102.48	0.002	0.018	–

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 1$ year or 3 years
Risk-free rate: 5% per annum, continuously compounded
Volatilities: $\nu_1 = \nu_2 = 20\%$
Spot prices at t_0: $x_1(0) = x_2(0) = 50$
Long-run mean prices: $\gamma_1 = \gamma_2 = 50e^{5\%T}$
Strength of mean-reversion: $\kappa_1 = \kappa_2 = \frac{1}{3}$
Time is in seconds

underlying assets with longer maturities and/or higher volatilities. Representative results for increasing maturity are shown in Table 5 while those for higher volatility and higher dimensionality are presented in Tables 6–8.

At the bottom of Table 5, we present the results for 3-year options on the maximum or minimum of two underlying assets with mean-reverting prices. Generally speaking, the relative errors do increase with time to maturity but the estimation accuracy is still acceptable. For in-the-money options, the highest relative errors increase to −0.048% for the call ($K = 40, \rho = 0.5$) and 0.323% for the put ($K = 55, \rho = 0.5$).

In Table 6, we present two-asset cases with higher volatilities. Fixing the time to maturity as 1 year and the correlation between the two underlying assets as 0.5, we test the proposed valuation approach using three volatilities: 60% volatility

Table 6 European put options on the minimum of and European call options on the maximum of two mean-reverting assets with high volatilities in the CIR Model

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Volatilities $\nu_1 = \nu_2 = 60\%$											
50	1.623(0.006)	34.40	1.612	0.018	−0.709	30	21.097(0.013)	34.40	21.095	0.018	−0.010
55	5.501(0.011)	34.40	5.508	0.018	0.124	35	16.340(0.014)	34.40	16.339	0.018	−0.004
60	10.204(0.009)	34.40	10.202	0.018	−0.023	40	11.592(0.010)	34.40	11.583	0.018	−0.081
65	14.957(0.011)	34.40	14.956	0.018	−0.004	45	6.849(0.010)	34.40	6.839	0.018	−0.142
70	19.716(0.009)	34.40	19.713	0.018	−0.020	50	2.568(0.008)	34.40	2.568	0.018	−0.002
75	24.468(0.013)	34.40	24.469	0.018	0.004	55	0.386(0.004)	34.40	0.401	0.018	3.821
Case 2: Volatilities $\nu_1 = 60\%$, $\nu_2 = 100\%$											
50	2.383(0.010)	34.40	2.418	0.018	1.458	30	21.723(0.017)	34.40	21.7130	0.018	−0.045
55	6.182(0.014)	34.40	6.236	0.018	0.888	35	16.962(0.020)	34.40	16.957	0.018	−0.033
60	10.832(0.012)	34.40	10.834	0.018	0.012	40	12.220(0.014)	34.40	12.202	0.018	−0.150
65	15.582(0.015)	34.40	15.575	0.018	−0.044	45	7.500(0.014)	34.40	7.501	0.018	0.002
70	20.342(0.012)	34.40	20.331	0.018	−0.056	50	3.341(0.012)	34.40	3.389	0.018	1.452
75	25.092(0.016)	34.40	25.087	0.018	−0.021	55	0.970(0.008)	34.40	0.957	0.018	−1.330
Case 3: Volatilities $\nu_1 = \nu_2 = 100\%$											
50	2.971(0.010)	34.40	2.937	0.018	−1.152	30	22.016(0.021)	34.40	22.008	0.018	−0.035
55	6.668(0.018)	34.40	6.667	0.018	−0.017	35	17.257(0.023)	34.40	17.252	0.018	−0.030
60	11.162(0.019)	34.40	11.163	0.018	0.011	40	12.525(0.017)	34.40	12.503	0.018	−0.178
65	15.885(0.016)	34.40	15.875	0.018	−0.062	45	7.908(0.016)	34.40	7.882	0.018	−0.333
70	20.633(0.016)	34.40	20.626	0.018	−0.034	50	3.952(0.014)	34.40	3.952	0.018	−0.013
75	25.390(0.016)	34.40	25.382	0.018	−0.033	55	1.425(0.009)	34.40	1.465	0.018	2.833

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 1$ year
Risk-free rate: 5% per annum, continuously compounded
Correlation: $\rho = 0.5$
Spot prices at t_0: $x_1(0) = x_2(0) = 50$
Long-run mean prices: $\gamma_1 = \gamma_2 = 50e^{5\%T}$
Strength of mean-reversion: $\kappa_1 = \kappa_2 = \frac{1}{3}$
Time is in seconds

for both assets (Case 1), 60% for the first asset and 100% for the second one (Case 2), and 100% for both (Case 3).[8] Generally speaking, when the underlying assets become more volatile, the relative errors are higher. Premiums for out-of-the-money options and at-the-money options, including both call on maximum and put on minimum options, are relatively small; therefore, the relative errors are relatively large accordingly. For in-the-money options, the largest relative error is 0.888% occurring when the strike price of a minimum put option is $55 in Case 2.

The tests for higher dimensionality are presented in Table 7 for four assets and in Table 8 for ten assets. For both sets of results, the initial prices for the assets are all $50, the pairwise correlations are 0.5, the strength of mean-reversion equal to $\frac{1}{3}$, and the continuously compounded risk-free rate is 5% per annum. Four sub-cases with different combinations of option type (minimum put options or maximum call

[8] The authors thank an anonymous referee for pointing out that, in the case of energy prices, the volatility can be as high as 100%.

Table 7 European put options on the minimum of and European call options on the maximum of four mean-reverting assets in the CIR model

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Volatilities $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 20\%$											
50	0.472(0.002)	67.42	0.466	0.018	−1.240	30	20.560(0.003)	67.42	20.561	0.018	0.002
55	4.904(0.003)	67.42	4.903	0.018	−0.031	35	15.804(0.003)	67.42	15.805	0.018	0.004
60	9.658(0.003)	67.42	9.659	0.018	0.005	40	11.047(0.003)	67.42	11.049	0.018	0.012
65	14.414(0.004)	67.42	14.415	0.018	0.005	45	6.294(0.003)	67.42	6.292	0.018	−0.004
70	19.172(0.004)	67.42	19.171	0.018	−0.006	50	1.561(0.004)	67.42	1.562	0.018	0.060
Case 2: Volatilities $\nu_1 = 20\%,\ \nu_2 = \nu_3 = 40\%,\ \nu_4 = 30\%$											
50	1.102(0.004)	67.42	1.099	0.018	−0.271	30	21.184(0.006)	67.42	21.192	0.018	0.040
55	5.520(0.005)	67.42	5.523	0.018	0.053	35	16.427(0.005)	67.42	16.436	0.018	0.054
60	10.273(0.005)	67.42	10.279	0.018	0.062	40	11.671(0.005)	67.42	11.680	0.018	0.079
65	15.029(0.007)	67.42	15.035	0.018	0.042	45	6.917(0.005)	67.42	6.924	0.018	0.099
70	19.788(0.007)	67.42	19.791	0.018	0.017	50	2.218(0.007)	67.42	2.241	0.018	1.041

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 1$ year
Risk-free rate: 5% per annum, continuously compounded
Correlation: $\rho = (\rho_{ij})_{4\times 4}$, where $\rho_{ij} = 1$ (if $i = j$), $\rho_{ij} = 0.5$ (otherwise)
Spot prices at t_0: $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 50$
Long-run mean prices: $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 50e^{5\%T}$
Strength of mean-reversion: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \frac{1}{3}$
Time is in seconds

options) and volatilities[9] are presented. The largest relative error of in-the-money options is 0.099% when the strike price of the maximum call option on underlying assets with different volatilities is \$45.

In Table 8 in which ten-asset cases are presented, we include six sub-cases with different combinations of option types and volatilities. As the deviation between estimated results and benchmarks increases in the volatilities of underlying assets, the largest relative error for in-the-money options is 0.204% occurring when the strike price of the maximum call option with the highest volatilities 60% is \$45.

In terms of computational efficiency, the time needed by one round of the Monte-Carlo simulation approach increases from 34.40 s for two assets to 67.42 s for four assets and to 167.48 s for ten assets. The computational time for the proposed approach remains at 0.018 s.

3.3 The effects of time-dependent correlations

Correlations between assets may not be fixed over time. Relaxing this assumption, we model the time-dependent correlation as $\rho(t) = a\sin(2\pi t) + b$, where a and b are constant and $\rho(t) \in [-1, 1]$. The same correlation function is assumed for all pairs of assets.

[9] $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 20\%$ or $\nu_1 = 20\%$, $\nu_2 = \nu_3 = 40\%$, $\nu_4 = 30\%$.

Table 8 European put options on the minimum of and European call options on the maximum of ten mean-reverting assets in the CIR model

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Volatilities $\nu_i = 20\%$ for $i = 1, \ldots, 10$											
50	0.717(0.003)	167.48	0.704	0.018	−1.734	30	20.980(0.004)	167.48	20.981	0.018	0.004
60	10.072(0.004)	167.48	10.064	0.018	−0.081	35	16.224(0.003)	167.48	16.225	0.018	0.007
65	14.827(0.003)	167.48	14.820	0.018	−0.051	40	11.468(0.003)	167.48	11.469	0.018	0.010
70	19.583(0.004)	167.48	19.576	0.018	−0.034	45	6.712(0.004)	167.48	6.713	0.018	0.011
75	24.340(0.002)	167.48	24.332	0.018	−0.031	50	1.964(0.003)	167.48	1.963	0.018	−0.037
Case 2: Volatilities $\nu_i = 20\%$ for $i = 1, \ldots, 5$ and $\nu_i = 40\%$ for $i = 6, \ldots, 10$											
50	1.561(0.005)	167.48	1.569	0.018	0.575	30	21.858(0.007)	167.48	21.869	0.018	0.049
60	10.926(0.007)	167.48	10.913	0.018	−0.115	35	17.105(0.006)	167.48	17.112	0.018	0.041
65	15.680(0.006)	167.48	15.669	0.018	−0.070	40	12.347(0.005)	167.48	12.356	0.018	0.076
70	20.439(0.005)	167.48	20.425	0.018	−0.067	45	7.591(0.004)	167.48	7.600	0.018	0.117
75	25.196(0.006)	167.48	25.181	0.018	−0.057	50	2.845(0.005)	167.48	2.873	0.018	0.982
Case 3: Volatilities $\nu_i = 20\%$ for $i = 1, \ldots, 4$, $\nu_i = 40\%$ for $i = 5, \ldots, 8$, $\nu_9 = \nu_{10} = 60\%$											
50	2.109(0.008)	167.48	2.147	0.018	1.811	30	22.425(0.007)	167.48	22.443	0.018	0.079
60	11.473(0.008)	167.48	11.452	0.018	−0.186	35	17.672(0.008)	167.48	17.687	0.018	0.084
65	16.227(0.006)	167.48	16.208	0.018	−0.120	40	12.915(0.009)	167.48	12.930	0.018	0.123
70	20.985(0.007)	167.48	20.964	0.018	−0.099	45	8.158(0.008)	167.48	8.174	0.018	0.204
75	25.742(0.006)	167.48	25.720	0.018	−0.083	50	3.413(0.006)	167.48	3.475	0.018	1.813

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 1$ year
Risk-free rate: 5% per annum, continuously compounded
Correlation: $\rho = (\rho_{ij})_{10\times 10}$, where $\rho_{ij} = 1$ (if $i = j$), $\rho_{ij} = 0.5$ (otherwise)
Spot prices at t_0: $x_i(0) = 50$ for $i = 1, \ldots, 10$
Long-run mean prices: $\gamma_i = 50e^{5\%T}$ for $i = 1, \ldots, 10$
Strength of mean-reversion: $\kappa_i = \frac{1}{3}$ for $i = 1, \ldots, 10$
Time is in seconds

In Tables 9 and 10, we present representative results for options on two and four underlying assets, respectively, with various time-dependent correlations and two different maturities. These examples are again for the CIR model of mean-reversion. In all cases, the spot prices of underlying assets are set equal to \$50, the volatilities to 40%, the strength of mean-reversion to $\frac{1}{3}$, and the continuously compounded risk-free rate to 5% per annum. The results truly highlight the computational efficiency of the proposed approximation approach. The time needed for the proposed approach remains at 0.018 s while that for the Monte-Carlo simulations increases a lot.

We present three sub-cases, with a correlation $0.3\sin(2\pi t)$, $0.6\sin(2\pi t) + 0.1$, $0.9\sin(2\pi t) + 0.05$, respectively, in Table 9 for both maximum call and minimum put options on two underlying assets with a 3-year time to maturity. We find that, generally speaking, the relative error between estimated results and benchmarks becomes larger when the range of correlations is larger. For in-the-money options, the largest relative error is 1.498% when the strike price of the minimum put option is \$55 with a correlation $0.9\sin(2\pi t) + 0.05$. The results of an at-the-money maximum call option with a strike price is also reported in Table 9 as its option price is

Table 9 European put options on the minimum of and European call options on the maximum of two mean-reverting assets in the CIR model under time-dependent correlations

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Correlation $\rho(t) = 0.3\sin(2\pi t)$											
55	1.865(0.006)	102.48	1.880	0.018	0.843	30	23.270(0.007)	102.48	23.236	0.018	−0.146
60	5.814(0.009)	102.48	5.831	0.018	0.288	35	18.965(0.007)	102.48	18.933	0.018	−0.169
65	10.112(0.008)	102.48	10.128	0.018	0.158	40	14.665(0.007)	102.48	14.629	0.018	−0.243
70	14.413(0.009)	102.48	14.431	0.018	0.128	45	10.358(0.009)	102.48	10.326	0.018	−0.311
75	18.717(0.010)	102.48	18.735	0.018	0.093	50	6.057(0.008)	102.48	6.025	0.018	−0.522
Case 2: Correlation $\rho(t) = 0.6\sin(2\pi t) + 0.1$											
55	1.816(0.006)	102.48	1.838	0.018	1.213	30	23.215(0.007)	102.48	23.153	0.018	−0.270
60	5.716(0.007)	102.48	5.750	0.018	0.595	35	18.910(0.008)	102.48	18.850	0.018	−0.319
65	10.010(0.008)	102.48	10.045	0.018	0.348	40	14.610(0.006)	102.48	14.546	0.018	−0.435
70	14.316(0.009)	102.48	14.348	0.018	0.223	45	10.303(0.009)	102.48	10.243	0.018	−0.588
75	18.618(0.008)	102.48	18.652	0.018	0.179	50	6.003(0.008)	102.48	5.943	0.018	−0.998
Case 3: Correlation $\rho(t) = 0.9\sin(2\pi t) + 0.05$											
55	1.832(0.006)	102.48	1.859	0.018	1.498	30	23.283(0.007)	102.48	23.195	0.018	−0.376
60	5.741(0.007)	102.48	5.791	0.018	0.863	35	18.977(0.008)	102.48	18.892	0.018	−0.449
65	10.037(0.010)	102.48	10.087	0.018	0.498	40	14.675(0.005)	102.48	14.588	0.018	−0.595
70	14.342(0.007)	102.48	14.390	0.018	0.334	45	10.370(0.009)	102.48	10.285	0.018	−0.825
75	18.646(0.009)	102.48	18.694	0.018	0.254	50	6.070(0.007)	102.48	5.985	0.018	−1.400

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 3$ years
Risk-free rate: 5% per annum, continuously compounded
Volatilities: $\nu_1 = \nu_2 = 40\%$
Spot prices at t_0: $x_1(0) = x_2(0) = 50$
Long-run mean prices: $\gamma_1 = \gamma_2 = 50e^{5\%T}$
Strength of mean-reversion: $\kappa_1 = \kappa_2 = \frac{1}{3}$
Time is in seconds

not very close to zero, and the highest relative error under different time-dependent correlations is -1.400% when the correlation is $0.9\sin(2\pi t) + 0.05$.

In Table 10, we present results from four-asset cases with different time to maturity by selecting the time-dependent correlation $0.3\sin(2\pi t)$. The time to maturity is either 1 year or 3 years. Other parameter values are the same as those in Table 9. The highest relative error for in-the-money options is 0.607% when the strike price of the minimum put option is \$55 and the time to maturity is 3 years, while that for at-the-money maximum call options is -0.744% as their premiums are small.

3.4 Further discussion

The most important observations about the above results on assets with mean-reverting processes are related to the computational efficiency and estimate accuracy. An important difference between this approach and methods based on Monte-Carlo simulations is that there is no trade-off between accuracy and computation time. This is an advantage as well as a disadvantage. The advantage is that increasing complexity and dimensionality does not increase the computation time. The

Table 10 European put options on the minimum of and European call options on the maximum of four mean-reverting assets in the CIR model under time-dependent correlations

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: Correlation $\rho(t) = 0.3\sin(2\pi t)$ and $T = 1$ year											
55	6.403(0.004)	67.42	6.414	0.018	0.169	30	22.153(0.005)	67.42	22.123	0.018	−0.134
60	11.160(0.006)	67.42	11.170	0.018	0.086	35	17.394(0.005)	67.42	17.367	0.018	−0.154
65	15.915(0.004)	67.42	15.926	0.018	0.067	40	12.639(0.005)	67.42	12.611	0.018	−0.222
70	20.671(0.005)	67.42	20.682	0.018	0.055	45	7.882(0.005)	67.42	7.855	0.018	−0.344
75	25.428(0.005)	67.42	25.438	0.018	0.042	50	3.139(0.006)	67.42	3.115	0.018	−0.744
Case 2: Correlation $\rho(t) = 0.3\sin(2\pi t)$ and $T = 3$ years											
55	2.856(0.005)	201.18	2.873	0.018	0.607	30	24.661(0.006)	201.18	24.623	0.018	−0.156
60	7.101(0.007)	201.18	7.118	0.018	0.241	35	20.358(0.007)	201.18	20.319	0.018	−0.192
65	11.407(0.007)	201.18	11.422	0.018	0.128	40	16.053(0.005)	201.18	16.016	0.018	−0.235
70	15.710(0.006)	201.18	15.725	0.018	0.097	45	11.750(0.005)	201.18	11.712	0.018	−0.321
75	20.015(0.007)	201.18	20.029	0.018	0.068	50	7.447(0.0060)	201.18	7.409	0.018	−0.511

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 1$ year or 3 years
Risk-free rate: 5% per annum, continuously compounded
Volatilities: $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 40\%$
Spot prices at t_0: $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 50$
Long-run mean prices: $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 50e^{5\%T}$
Strength of mean-reversion: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \frac{1}{3}$
Time is in seconds

disadvantage is that the accuracy cannot be improved by increasing the computation time.

4 Applications to assets with stochastic volatilities

Most previous studies of maximum-minimum options assume constant or deterministic volatilities. Many applications may be on assets for which this assumption does not hold. In this section, we extend the applicability of the proposed approach to maximum–minimum options on assets with stochastic volatilities.

One of the most widely accepted process with stochastic volatility is the one proposed by Heston (1993). Following Heston's setting, we assume that each asset x_i follows the process

$$\begin{cases} dx_i(t) = x_i(t)\{rdt + \sqrt{V_i(t)}dz_i(t)\}, \\ x_i(0) = x_{i0}, \end{cases} \tag{16}$$

where the volatility V_i satisfies

$$\begin{cases} dV_i(t) = \kappa_i(\theta_i - V_i(t))dt + \sigma\sqrt{V_i(t)}dz_{iv}(t), \\ V_i(0) = V_{i0}. \end{cases} \tag{17}$$

For analytical simplicity, we follow Hull (2006) and assume that $dz_i(t)$ and $dz_{iv}(t)$ are uncorrelated.

To adjust the semi-analytic approach for underlying assets with stochastic volatilities, let $g_i = \ln x_i$, by Itô's formula,

$$\begin{cases} dg_i(t) = (r - \frac{1}{2}V_i(t))dt + \sqrt{V_i(t)}dz_i(t), \\ g_i(0) = \ln x_{i0}. \end{cases} \tag{18}$$

Taking expectations of both sides of (18) and those of (17) and integrating, we obtain the first moment

$$\hat{E}g_i(t) = (r - \frac{1}{2}\theta_i)t + \frac{1}{2\kappa_i}(V_{i0} - \theta_i)(e^{-\kappa_i t} - 1) + \ln x_{i0}, \quad \forall t \in [0, T].$$

Applying the Ito's Lemma to $g_i(t)^2$ and taking expectations and integrating, we also obtain the second moment.

$$\begin{aligned} \text{Var } g_i(t) &= \hat{E}(g_i(t) - \hat{E}g_i(t))^2 \\ &= \int_0^t (-\hat{E}(g_i(s)V_i(s)) + \hat{E}g_i(s)\hat{E}V_i(s) + \hat{E}V_i(s))ds \\ &= \int_0^t [-\hat{E}(g_i(s)V_i(s)) + (\hat{E}g_i(s) + 1)\hat{E}V_i(s)]ds \\ &= \frac{\sigma_i^2}{2\kappa_i^2}(\theta_i - V_{i0})te^{-\kappa_i t} + [\frac{\sigma_i^2}{2\kappa_i^3}\theta_i + \frac{1}{\kappa_i}(\theta_i - V_{i0})]e^{-\kappa_i t} \\ &\quad + \frac{\sigma_i^2}{4\kappa_i^3}(\frac{1}{2}\theta_i - V_{i0})e^{-2\kappa_i t} + \theta_i(\frac{\sigma_i^2}{4\kappa_i^2} + 1)t \\ &\quad - [\frac{1}{\kappa_i^2}(\frac{5\sigma_i^2}{8\kappa_i}\theta_i - \frac{\sigma_i^2}{4\kappa_i}V_{i0}) + \frac{1}{\kappa_i}(\theta_i - V_{i0})]. \end{aligned}$$

After deriving the first and second moments based on Heston's model with stochastic volatility (see Appendix), we apply the two-step semi-analytic approach via a lognormal process to valuate high-dimensional options on the maximum and minimum of multiple underlying assets with stochastic volatilities. Given that

$$\begin{cases} \hat{E}x_i(t) = \exp(\hat{E}g_i(t) + \frac{1}{2}\text{Var } g_i(t)), \\ \hat{E}x_i(t)^2 = (\hat{E}x_i(t))^2 \exp(\text{Var } g_i(t)), \end{cases}$$

we approximate the high-dimensional option via an artificial lognormal process, and then apply the Black–Scholes model to obtain the option price.

In Table 11, we present cases for high-dimensional call options on the maximum and put options on the minimum of four underlying assets with stochastic volatilities following Heston's (1993) model. Most of the parameter values are the same as those in the cases presented in Section 3. The spot prices of the four underlying assets are \$50, the continuously compounded risk-free rate is 5% per annum, and the correlations between any pair of the four underlying assets are equal to 0.5. Their initial volatilities are the same (0.2) and the long-run means of their stochastic volatilities are also the same (0.1). For analytical simplicity, we assume that the Brownian motion of an underlying asset and that of its stochastic volatility are uncorrelated. The strength of mean-reversion of the stochastic volatility

Table 11 European put options on the minimum of and European call options on the maximum of four assets with stochastic volatility following Heston's (1993) model

Minimum put						Maximum call					
K	MC (SD)	Time	LW	Time	Difference (%)	K	MC (SD)	Time	LW	Time	Difference (%)
Case 1: $T = 3$ months											
40	1.701(0.008)	44.16	1.666	0.018	−2.063	45	13.806(0.034)	44.16	13.834	0.018	0.204
45	4.145(0.016)	44.16	4.144	0.018	−0.041	50	9.566(0.028)	44.16	9.602	0.018	0.376
50	7.718(0.025)	44.16	7.744	0.018	0.337	55	6.130(0.018)	44.16	6.149	0.018	0.300
55	12.045(0.022)	44.16	12.073	0.018	0.236	60	3.625(0.016)	44.16	3.628	0.018	0.086
60	16.749(0.024)	44.16	16.771	0.018	0.131	65	1.990(0.012)	44.16	1.981	0.018	−0.417
65	21.610(0.024)	44.16	21.624	0.018	0.068	70	1.021(0.011)	44.16	1.009	0.018	−1.165
70	26.505(0.027)	44.16	26.535	0.018	0.113	75	0.497(0.008)	44.16	0.484	0.018	−2.663
Case 2: $T = 1$ year											
40	5.267(0.019)	134.18	5.272	0.018	0.103	45	22.655(0.061)	134.18	22.666	0.018	0.047
45	8.386(0.024)	134.18	8.411	0.018	0.294	50	18.753(0.069)	134.18	18.737	0.018	−0.087
50	12.045(0.042)	134.18	12.095	0.018	0.413	55	15.231(0.070)	134.18	15.234	0.018	0.021
55	16.115(0.026)	134.18	16.166	0.018	0.314	60	12.220(0.046)	134.18	12.200	0.018	−0.163
60	20.425(0.043)	134.18	20.495	0.018	0.343	65	9.706(0.053)	134.18	9.639	0.018	−0.687
65	24.932(0.037)	134.18	24.990	0.018	0.236	70	7.603(0.056)	134.18	7.526	0.018	−1.019
70	29.551(0.027)	134.18	29.589	0.018	0.131	75	5.887(0.030)	134.18	5.816	0.018	−1.200
Case 3: $T = 3$ years											
40	8.366(0.026)	410.48	8.394	0.018	0.336	45	36.235(0.142)	410.48	36.057	0.018	−0.490
45	11.442(0.031)	410.48	11.484	0.018	0.362	50	32.714(0.115)	410.48	32.544	0.018	−0.520
50	14.824(0.034)	410.48	14.871	0.018	0.315	55	29.433(0.127)	410.48	29.276	0.018	−0.534
55	18.421(0.035)	410.48	18.484	0.018	0.346	60	26.504(0.101)	410.48	26.261	0.018	−0.917
60	22.219(0.036)	410.48	22.270	0.018	0.228	65	23.681(0.131)	410.48	23.502	0.018	−0.753
65	26.134(0.052)	410.48	26.184	0.018	0.189	70	21.214(0.109)	410.48	20.993	0.018	−1.042
70	30.114(0.044)	410.48	30.194	0.018	0.263	75	18.912(0.089)	410.48	18.723	0.018	−1.000

The column indexed by LW indicates the option prices obtained from the method proposed in the current study, while those indexed by MC present option prices from Monte-Carlo simulations. The relative error is measured by the difference between results obtained from these two methods
Time to maturity: $T = 0.25$, 1 year or 3 years
Risk-free rate: 5% per annum
Spot prices at t_0: $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 50$
Correlation: $\rho = (\rho_{ij})_{10\times 10}$, where $\rho_{ij} = 1$ (if $i = j$), $\rho_{ij} = 0.5$ (otherwise)
Parameters: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 10\%$
Initial volatility: $V_1(0) = V_2(0) = V_3(0) = V_4(0) = 0.2$
Long-run mean of volatility: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.1$
Strength of mean-reversion of volatility: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 2.0$
Time is in seconds

of each underlying asset is 2.0. Three different times to maturity: 3 months, 1 year, and 3 years were tested.

Besides in-the-money options as usual, both at-the-money and out-of-the-money options are also adopted for the simulations. Relative errors between the simulation results from the proposed approximation approach and the benchmarks generated from Monte-Carlo simulations are fairly small. The highest relative error is −2.663% which occurs in an out-of-the-money maximum call option with a strike price of $75 and a 3-month time to maturity. These again illustrate the estimation accuracy of the proposed approach. The savings in computation time is practically 100% compared with using Monte-Carlo simulation for the 3-year option.

5 Concluding remarks

The trade-off between accuracy and computational efficiency when numerical methods are used to value options is well known; the computation time needed to improve accuracy increases significantly. This article presents a new simple but powerful two-step approach for pricing high-dimensional European options on the maximum or the minimum of multiple underlying assets. Numerical examples show that the procedure is at least as accurate as extant methods but the computation time is reduced to practically zero.

The approximation procedure is shown to be applicable to high-dimensional options on the maximum or minimum of assets with unknown distributions, mean-reverting processes, and stochastic volatilities. Moreover, it is shown to be robust for high volatilities, time-varying correlations among asset, long maturities, and high dimensionalities. While accuracy deteriorates slightly with increasing complexity of the option, in comparison with results from Monte-Carlo simulations, it remains within very acceptable range. The greatest advantage of the approach is that the computation time does not increase with the complexity of the option.

The proposed approach adds to the literature on option valuation when the underlying asset's stochastic process is not lognormal or unknown. It is easy to implement and, thus, has the potential of being widely adopted by academics and practitioners.

We have not tested the approach on more complicated processes, for example, those with jumps. This should be the subject of future research in order to gain a better understanding of the limitations of the approach.

Appendix A: the first and second moments in Section 4

Following Heston's setting, we assume that an asset x_i follows the process (16) and its volatility V_i satisfies (17). Let $g_i = \ln x_i$. When $dz_i(t)$ and $dz_{iv}(t)$ are uncorrelated, applying Itô's formula yields (18).

Taking expectation of both sides of (17) yields

$$\begin{cases} d\hat{E}V_i(t) = \kappa_i(\theta_i - \hat{E}V_i(t))dt, \\ \hat{E}V_i(0) = V_{i0}, \end{cases} \tag{19}$$

Integrating (19) yields

$$\begin{aligned} \hat{E}V_i(t) &= V_{i0}e^{-\kappa_i t} - \theta_i(e^{-\kappa_i t} - 1) \\ &= (V_{i0} - \theta_i)e^{-\kappa_i t} + \theta_i. \end{aligned} \tag{20}$$

Taking expectation of both sides of (18) yields

$$\begin{cases} d\hat{E}g_i(t) = (r - \frac{1}{2}\hat{E}V_i(t))dt, \\ \hat{E}g_i(0) = \ln x_{i0}. \end{cases} \tag{21}$$

Integrating (21) yields

$$\hat{E}g_i(t) = rt - \frac{1}{2}\int_0^t \hat{E}V_i(s)ds + \ln x_{i0}. \tag{22}$$

Substituting (20) into (22) yields

$$\hat{E}g_i(t) = (r - \frac{1}{2}\theta_i)t + \frac{1}{2\kappa_i}(V_{i0} - \theta_i)(e^{-\kappa_i t} - 1) + \ln x_{i0}. \tag{23}$$

Applying Itô's formula to $V_i(t)^2$ yields

$$\begin{cases} dV_i(t)^2 = 2V_i(t)[\kappa_i(\theta_i - V_i(t))dt + \sigma_i\sqrt{V_i(t)}dz_{iv}(t)] + \sigma_i^2 V_i(t)dt \\ \qquad\quad = [(2\kappa_i\theta_i + \sigma_i^2)V_i(t) - 2\kappa_i V_i(t)^2]dt + 2\sigma_i V_i(t)\sqrt{V_i(t)}dz_{iv}(t), \\ V_i(0)^2 = V_{i0}^2, \end{cases} \tag{24}$$

Taking expectation of both sides of (24) yields

$$\begin{cases} d\hat{E}V_i(t)^2 = [(2\kappa_i\theta_i + \sigma_i^2)\hat{E}V_i(t) - 2\kappa_i\hat{E}V_i(t)^2]dt, \\ \hat{E}V_i(0)^2 = V_{i0}^2. \end{cases} \tag{25}$$

Hence,

$$\hat{E}V_i(t)^2 = [V_{i0}^2 + (2\kappa_i\theta_i + \sigma_i^2)\int_0^t \hat{E}V_i(s)e^{2\kappa_i s}ds]e^{-2\kappa_i t}. \tag{26}$$

Substituting (20) into (26) yields

$$\begin{aligned} \hat{E}V_i(t)^2 = V_{i0}^2 e^{-2\kappa_i t} + \frac{2\kappa_i\theta_i + \sigma_i^2}{\kappa_i}[(\frac{1}{2}\theta_i - V_{i0})e^{-2\kappa_i t} \\ +(V_{i0} - \theta_i)e^{-\kappa_i t} + \frac{1}{2}\theta_i]. \end{aligned} \tag{27}$$

Based on (17) and (18), applying Itô's formula to $g_i(t)V_i(t)$ yields

$$\begin{cases} d(g_i(t)V_i(t)) = \frac{\partial(g_i(t)V_i(t))}{\partial g_i(t)}dg_i(t) + \frac{\partial(g_i(t)V_i(t))}{\partial V_i(t)}dV_i(t) \\ \qquad\qquad + \frac{\partial^2(g_i(t)V_i(t))}{\partial g_i(t)\partial V_i(t)}\sqrt{V_i(t)}\sigma_i\sqrt{V_i(t)}dz_i(t)dz_{iv}(t) \\ \qquad\qquad = V_i(t)[(r - \frac{1}{2}V_i(t))dt + \sqrt{V_i(t)}dz_i(t)] \\ \qquad\qquad + g_i(t)[\kappa_i(\theta_i - V_i(t))dt + \sigma_i\sqrt{V_i(t)}dz_{iv}(t)] \\ \qquad\qquad + \sqrt{V_i(t)}\sigma_i\sqrt{V_i(t)} \times 0 \times dt \\ \qquad\qquad = (rV_i(t) - \frac{1}{2}V_i(t)^2 + \kappa_i\theta_i g_i(t) - \kappa_i g_i(t)V_i(t))dt \\ \qquad\qquad + V_i(t)\sqrt{V_i(t)}dz_i(t) + \sigma_i g_i(t)\sqrt{V_i(t)}dz_{iv}(t), \\ g_i(0)V_i(0) = V_{i0}\ln x_{i0}. \end{cases} \tag{28}$$

Since

$$\begin{cases} d\hat{E}(g_i(t)V_i(t)) = (r\hat{E}V_i(t) - \frac{1}{2}\hat{E}V_i(t)^2 + \kappa_i\theta_i\hat{E}g_i(t) \\ \qquad\qquad - \kappa_i\hat{E}(g_i(t)V_i(t)))dt, \\ \hat{E}(g_i(0)V_i(0)) = V_0\ln x_0, \end{cases} \tag{29}$$

we have

$$\begin{aligned} \hat{E}(g_i(t)V_i(t)) = [V_{i0}\ln x_{i0} + \int_0^t (r\hat{E}V_i(s) - \frac{1}{2}\hat{E}V_i(s)^2 \\ +\kappa_i\theta_i\hat{E}g_i(s))e^{\kappa_i s}ds]e^{-\kappa_i t}. \end{aligned} \tag{30}$$

Substituting (20), (23) and (27) into (30), we have

$$
\begin{aligned}
\hat{E}(g_i(t)V_i(t)) = & \Big\{V_{i0}\ln x_{i0} + \int_0^t r[(V_{i0}-\theta_i)e^{-\kappa_i s} + \theta_i]e^{\kappa_i s}ds \\
& -\frac{1}{2}\int_0^t \Big\{V_{i0}^2 e^{-2\kappa_i s} + \frac{2\kappa_i\theta_i + \sigma_i^2}{\kappa_i}[(\frac{1}{2}\theta_i - V_{i0})e^{-2\kappa_i s} \\
& +(V_{i0}-\theta_i)e^{-\kappa_i s} + \frac{1}{2}\theta_i]\Big\} e^{\kappa_i s}ds \\
& +\kappa_i\theta_i \int_0^t [(r-\frac{1}{2}\theta_i)s + \frac{1}{2\kappa_i}(V_{i0}-\theta_i)(e^{-\kappa_i s}-1) \\
& +\ln x_{i0}]e^{\kappa_i s}ds\Big\} e^{-\kappa_i t} \\
= & \Big\{V_{i0}\ln x_{i0} + \int_0^t r[(V_{i0}-\theta_i) + \theta_i e^{\kappa_i s}]ds \\
& -\frac{1}{2}\int_0^t \Big\{V_{i0}^2 e^{-\kappa_i s} + \frac{2\kappa_i\theta_i + \sigma_i^2}{\kappa_i}[(\frac{1}{2}\theta_i - V_{i0})e^{-\kappa_i s} + (V_{i0}-\theta_i) \\
& +\frac{1}{2}\theta_i e^{\kappa_i s}]\Big\} ds + \kappa_i\theta_i \int_0^t [(r-\frac{1}{2}\theta_i)se^{\kappa_i s} + \frac{1}{2\kappa_i}(V_{i0}-\theta_i) \\
& +(\ln x_{i0} - \frac{1}{2\kappa_i}(V_{i0}-\theta_i))e^{\kappa_i s}]ds\Big\} e^{-\kappa_i t} \\
= & (r - \frac{\theta_i}{2} - \frac{\sigma_i^2}{2\kappa_i})(V_{i0}-\theta_i)te^{-\kappa_i t} \\
& +[V_{i0}\ln x_{i0} - \theta_i(\frac{\theta_i}{\kappa_i} + \ln x_{i0} - \frac{3V_{i0}}{2\kappa_i}) - (\frac{V_{i0}^2}{2\kappa_i} - \frac{\sigma_i^2}{2\kappa_i^2}V_{i0})]e^{-\kappa_i t} \\
& +[\frac{V_{i0}^2}{2\kappa_i} + \frac{2\kappa_i\theta_i + \sigma_i^2}{2\kappa_i^2}(\frac{1}{2}\theta_i - V_{i0})]e^{-2\kappa_i t} + \theta_i(r - \frac{1}{2}\theta_i)t \\
& +\theta_i(\frac{\theta_i}{2\kappa_i} - \frac{\sigma_i^2}{4\kappa_i^2} + \ln x_{i0} - \frac{V_{i0}}{2\kappa_i}).
\end{aligned}
\tag{31}
$$

From (18) and (21), we have

$$
\begin{cases}
d(g_i(t) - \hat{E}g_i(t)) = -\frac{1}{2}(V_i(t) - \hat{E}V_i(t))dt + \sqrt{V_i(t)}dz_i(t), \\
g_i(0) - \hat{E}g_i(0) = 0.
\end{cases}
\tag{32}
$$

Applying Itô's formula to $(g_i(t) - \hat{E}g_i(t))^2$ yields

$$
\begin{cases}
d(g_i(t) - \hat{E}g_i(t))^2 = 2(g_i(t) - \hat{E}g_i(t))d(g_i(t) - \hat{E}g_i(t)) \\
\qquad\qquad +[d(g_i(t) - \hat{E}g_i(t))]^2 \\
\qquad = [-(g_i(t) - \hat{E}g_i(t))(V_i(t) - \hat{E}V_i(t)) + V_i(t)]dt \\
\qquad\qquad +2(g_i(t) - \hat{E}g_i(t))\sqrt{V_i(t)}dz_i(t)\}, \\
(g_i(0) - \hat{E}g_i(0))^2 = 0.
\end{cases}
\tag{33}
$$

Furthermore,

$$\begin{cases} d\hat{E}(g_i(t)-\hat{E}g_i(t))^2 = \hat{E}[-(g_i(t)-\hat{E}g_i(t))(V_i(t)-\hat{E}V_i(t))+V_i(t)]dt \\ \qquad\qquad\qquad\qquad\quad = (-\hat{E}(g_i(t)V_i(t))+\hat{E}g_i(t)\hat{E}V_i(t) \\ \qquad\qquad\qquad\qquad\qquad +\hat{E}V_i(t))dt, \\ \hat{E}(g_i(0)-\hat{E}g_i(0))^2 = 0. \end{cases} \tag{34}$$

Hence,

$$\begin{aligned} \mathrm{Var}\, g_i(t) &= \hat{E}(g_i(t)-\hat{E}g_i(t))^2 \\ &= \int_0^t (-\hat{E}(g_i(s)V_i(s))+\hat{E}g_i(s)\hat{E}V_i(s)+\hat{E}V_i(s))ds \\ &= \int_0^t [-\hat{E}(g_i(s)V_i(s))+(\hat{E}g_i(s)+1)\hat{E}V_i(s)]ds \\ &= -\int_0^t \{(r-\frac{\theta_i}{2}-\frac{\sigma_i^2}{2\kappa_i})(V_{i0}-\theta_i)se^{-\kappa_i s} \\ &\quad +[V_{i0}\ln x_{i0}-\theta_i(\frac{\theta_i}{\kappa_i}+\ln x_{i0}-\frac{3V_{i0}}{2\kappa_i})-(\frac{V_{i0}^2}{2\kappa_i}-\frac{\sigma_i^2}{2\kappa_i^2}V_{i0})]e^{-\kappa_i s} \\ &\quad +[\frac{V_{i0}^2}{2\kappa_i}+\frac{2\kappa_i\theta_i+\sigma_i^2}{2\kappa_i^2}(\frac{1}{2}\theta_i-V_{i0})]e^{-2\kappa_i s}+\theta_i(r-\frac{1}{2}\theta_i)s \\ &\quad +\theta_i(\frac{\theta_i}{2\kappa_i}-\frac{\sigma_i^2}{4\kappa_i^2}+\ln x_0-\frac{V_{i0}}{2\kappa_i})\}ds \\ &\quad +\int_0^t [(r-\frac{1}{2}\theta_i)s+\frac{1}{2\kappa_i}(V_{i0}-\theta_i)(e^{-\kappa_i s}-1)+\ln x_{i0}+1] \\ &\quad \times[(V_{i0}-\theta_i)e^{-\kappa_i s}+\theta_i]ds \\ &= \frac{\sigma_i^2}{2\kappa_i^2}(\theta_i-V_{i0})te^{-\kappa_i t}+\frac{\sigma_i^2}{2\kappa_i^3}\theta_i+\frac{1}{\kappa_i}(\theta_i-V_{i0})]e^{-\kappa_i t} \\ &\quad +\frac{\sigma_i^2}{4\kappa_i^3}(\frac{1}{2}\theta_i-V_{i0})e^{-2\kappa_i t}+\theta_i(\frac{\sigma_i^2}{4\kappa_i^2}+1)t \\ &\quad -[\frac{1}{\kappa_i^2}(\frac{5\sigma_i^2}{8\kappa_i}\theta_i-\frac{\sigma_i^2}{4\kappa_i}V_{i0})+\frac{1}{\kappa_i}(\theta_i-V_{i0})]. \end{aligned} \tag{35}$$

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